Online Appendix to Pooling and the Identification of Willingness to Pay

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Abstract

This document contains appendix material for Thakral and Tô (2026).

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A.1 Proof of Theorem 1

Proof. To prove the result about the median, we construct a distribution of B that holds the median bounded while moving b^p. Since F is strictly increasing with $\lim_{t\to-\infty} F(t) = 0$, we have $F(-M) > 0$. Choose $\lambda > \frac{1}{2}$ such that $1 - \lambda + \lambda F(-M) > \frac{1}{2}$. Let B equal 0 with probability λ and equal d with probability $1 - \lambda$. Then $\text{Med}[B] = 0$ because $\Pr(B \leq 0) = \lambda > \frac{1}{2}$. Moreover,

$$m(M) = \lambda F(-M) + (1 - \lambda)F(d - M).$$

Since $F(d - M) \to 1$ as $d \to \infty$, for sufficiently large d we have $m(M) > \lambda F(-M) + (1 - \lambda) - \eta > \frac{1}{2}$ for small enough $\eta > 0$. By monotonicity of m, $m(M) > \frac{1}{2}$ implies $b^p > M$, and therefore $|b^p - \text{Med}[B]| = b^p > M$.

To prove the result about the mean, we construct a distribution of B that holds b^p bounded while moving the mean. Let $\lambda \in (1/2, 1)$. For each $d > 0$, let B equal 0 with probability λ and equal d with probability $1 - \lambda$. Then $\mathbb{E}[B] = (1 - \lambda)d$. Define c by $F(-c) = 1 - \frac{1}{2\lambda}$, which is well-defined because $1 - \frac{1}{2\lambda} \in (0, 1)$ when $\lambda > 1/2$. For any $\varepsilon > 0$ and any $d > 0$,

$$m(c+\varepsilon) = \lambda F(-(c+\varepsilon))+(1-\lambda)F(d-c-\varepsilon) \leq \lambda F(-(c+\varepsilon))+(1-\lambda) < \lambda F(-c)+(1-\lambda) = \tfrac{1}{2},$$

so by strict monotonicity of m the crossing satisfies $b^p < c + \varepsilon$. Taking $\varepsilon = 1$, we obtain a constant $C = c + 1$ such that $b^p \leq C$ for all d. Hence

$$|b^p - \mathbb{E}[B]| \geq \mathbb{E}[B] - b^p \geq (1 - \lambda)d - C,$$

which exceeds M for d large enough.

If $\varepsilon = 0$, then

$$b^p = \text{Med}[B + \varepsilon] = \text{Med}[B]$$

for every distribution of B.

Conversely, suppose ε is nondegenerate. Let G denote its continuous, strictly increasing CDF, normalized so that

$$G(0) = \tfrac{1}{2}.$$

Fix any $d > 0$ and any $p \in (\frac{1}{2}, 1)$, and let

$$B = \begin{cases} 0 & \text{with probability } p, \\ d & \text{with probability } 1-p. \end{cases}$$

Then B has finite mean and

$$\text{Med}[B] = 0.$$

By independence, $F_{B+\varepsilon}(x) = pG(x) + (1-p)G(x-d)$. Evaluating at $x = 0$ gives $F_{B+\varepsilon}(0) = pG(0) + (1-p)G(-d)$. Since $G(0) = \frac{1}{2}$ and $G(-d) < \frac{1}{2}$, this implies $F_{B+\varepsilon}(0) < \frac{1}{2}$ and hence $\text{Med}[B+\varepsilon] > 0$. Thus, we conclude that $b^p = \text{Med}[B+\varepsilon] \neq 0 = \text{Med}[B]$. This establishes that $b^p = \text{Med}[B]$ for every distribution of B implies ε must be degenerate. Since ε has median zero, the only degenerate possibility is $\varepsilon = 0$. □

A.2 Proof of Corollary A.1

For any set $A \subseteq \mathbb{R}$, define its diameter by

$$\text{diam}(A) := \sup\{|x-y| : x, y \in A\}.$$

Corollary A.1. *For every $M > 0$, there exists an admissible pooled curve $m_E(\cdot)$ such that*

$$\text{diam}(\mathcal{I}_E(m_E)) > M,$$

and there exists an admissible pooled curve $m_M(\cdot)$ such that

$$\text{diam}(\mathcal{I}_M(m_M)) > M.$$

Proof. Fix $M > 0$.

For the mean identified set, by Theorem 1, there exists a data-generating process $(B_E, \varepsilon_E) \in \mathcal{M}$ with finite mean such that

$$|b_E^p - \mathbb{E}[B_E]| > M,$$

where b_E^p denotes its pooled crossing. Let $m_E(\cdot)$ denote the corresponding pooled curve, and let $W_E := B_E + \varepsilon_E$. Since $m_E(\Delta p) = \Pr(W_E \geq \Delta p)$, the pooled crossing satisfies $b_E^p = \text{Med}[W_E]$.

Now define an alternative model by

$$\tilde{B}_E \equiv b^p_E,$$
$$\tilde{\varepsilon}_E := W_E - b^p_E.$$

Because $\tilde{B}_E$ is degenerate, it is independent of $\tilde{\varepsilon}_E$. Since W_E has a continuous, strictly increasing CDF and $b^p_E = \text{Med}[W_E]$, the shock $\tilde{\varepsilon}_E$ has a continuous, strictly increasing CDF with median zero. Thus $(\tilde{B}_E, \tilde{\varepsilon}_E) \in \mathcal{M}$.

The alternative model generates the same pooled curve because, for each Δp,

$$\begin{aligned}
\Pr(\tilde{B}_E - \Delta p + \tilde{\varepsilon}_E \geq 0) &= \Pr(b^p_E - \Delta p + W_E - b^p_E \geq 0) \\
&= \Pr(W_E \geq \Delta p) \\
&= m_E(\Delta p).
\end{aligned}$$

Hence $\mathbb{E}[B_E] \in \mathcal{I}_E(m_E)$ by the original model, while $\mathbb{E}\left[\tilde{B}_E\right] = b^p_E \in \mathcal{I}_E(m_E)$ by the observationally equivalent point-mass model. Therefore

$$\text{diam}(\mathcal{I}_E(m_E)) \geq |\mathbb{E}[B_E] - b^p_E| > M.$$

For the median identified set, again by Theorem 1, there exists a data-generating process $(B_M, \varepsilon_M) \in \mathcal{M}$ such that

$$|b^p_M - \text{Med}[B_M]| > M,$$

where b^p_M denotes its pooled crossing. Let $m_M(\cdot)$ denote the corresponding pooled curve, and let $W_M := B_M + \varepsilon_M$. Since $m_M(\Delta p) = \Pr(W_M \geq \Delta p)$, the pooled crossing satisfies $b^p_M = \text{Med}[W_M]$.

Define an observationally equivalent point-mass model by

$$\tilde{B}_M \equiv b^p_M,$$
$$\tilde{\varepsilon}_M := W_M - b^p_M.$$

As above, $(\tilde{B}_M, \tilde{\varepsilon}_M) \in \mathcal{M}$ and this model generates the same pooled curve $m_M(\cdot)$. Hence $\text{Med}[B_M] \in \mathcal{I}_M(m_M)$ by the original model, while $\text{Med}\left[\tilde{B}_M\right] = b^p_M \in \mathcal{I}_M(m_M)$ by the observationally equivalent point-mass model. Therefore

$$\text{diam}(\mathcal{I}_M(m_M)) \geq |\text{Med}[B_M] - b^p_M| > M. \qquad \square$$

A.3 Theorem A.1 and its proof

Let $\mathcal{M}$ denote the class of data-generating processes such that

$$y = \mathbb{1}_{\{B - \Delta p + \varepsilon \geq 0\}},$$

where B is independent of ε and ε has a continuous, strictly increasing CDF with median zero. For any admissible pooled curve $m(\cdot)$, let $\mathcal{I}(m)$ denote the set of data-generating processes in $\mathcal{M}$ that satisfy

$$\Pr(B - \Delta p + \varepsilon \geq 0) = m(\Delta p)$$

for all Δp. Let

$$\mathcal{I}_E(m) := \{\mathbb{E}[B] : (B, \varepsilon) \in \mathcal{I}(m), \mathbb{E}[|B|] < \infty\}$$

and

$$\mathcal{I}_M(m) := \{\mathrm{Med}[B] : (B, \varepsilon) \in \mathcal{I}(m)\}.$$

Theorem A.1. *Let ψ be a real-valued functional of the marginal distribution of B. If there exists a mapping T from pooled curves $m(\cdot)$ to $\mathbb{R}$ such that*

$$T(m(\cdot)) = \psi(B)$$

for every data-generating process in $\mathcal{M}$, then ψ is constant over $\mathcal{M}$.

Proof. Write δ_c for the point mass at c.

Assume that ψ is identified from pooling, i.e. there exists a mapping T such that

$$T(m(\cdot)) = \psi(B)$$

for every data-generating process in $\mathcal{M}$.

Fix any data-generating process in $\mathcal{M}$. Because ε has a continuous, strictly increasing CDF, the CDF of $B + \varepsilon$ is also continuous and strictly increasing.[1] Let $c := \mathrm{Med}[B + \varepsilon]$. Now define an alternative model by $\tilde{B} \equiv c$ and $\tilde{\varepsilon} := B + \varepsilon - c$. Then $\tilde{B}$ is independent of $\tilde{\varepsilon}$; $\tilde{\varepsilon}$ has a continuous, strictly increasing CDF with median zero; and

$$\Pr(\tilde{B} - \Delta p + \tilde{\varepsilon} \geq 0) = \Pr(B + \varepsilon \geq \Delta p) = \Pr(B - \Delta p + \varepsilon \geq 0).$$

So the original model and the point-mass model generate the same pooled curve. By

[1] Indeed, if H denotes the CDF of $B + \varepsilon$, then for $x < y$, we have $H(y) - H(x) = \mathbb{E}[G(y - B) - G(x - B)] > 0$, where the inequality follows from strict monotonicity of G.

identification, $\psi(B) = \psi(\delta_c)$. Thus, for every model in $\mathcal{M}$, the value of $\psi(B)$ must equal the value of ψ at the point mass located at the pooled crossing.

We next show that $\psi(\delta_c)$ must in fact be the same for every $c \in \mathbb{R}$. Fix any $c_1 < c_2$. Choose numbers $a < c_1 < c_2 < d$, and let $\lambda \in (1/2, 1)$. Define a two-point distribution

$$B^* = \begin{cases} a & \text{with probability } \lambda, \\ d & \text{with probability } 1 - \lambda. \end{cases}$$

We claim that for each $r \in [c_1, c_2]$ there exists a shock distribution such that, when ε_r is drawn independently of B^*, the pooled crossing of the model (B^*, ε_r) equals r. To see this, fix any $r \in [c_1, c_2]$. Choose $u \in (1 - \frac{1}{2\lambda}, \frac{1}{2})$ and define

$$v := \frac{\frac{1}{2} - \lambda u}{1 - \lambda}.$$

Because $\lambda \in (1/2, 1)$, the interval for u is nonempty, and this choice gives $1/2 < v < 1$. Since $a < c_1 \leq r \leq c_2 < d$, we have $a - r < 0 < d - r$. Since also $u < 1/2 < v$, we can choose a continuous, strictly increasing CDF F_r on $\mathbb{R}$ such that $F_r(a - r) = u$, $F_r(0) = \frac{1}{2}$, and $F_r(d - r) = v$. Let ε_r be independent of B^* and have CDF $G_r(x) := 1 - F_r(-x)$. Then G_r is continuous and strictly increasing with median zero, so $(B^*, \varepsilon_r) \in \mathcal{M}$.

For this model, the pooled curve is

$$m_r(\Delta p) = \lambda F_r(a - \Delta p) + (1 - \lambda) F_r(d - \Delta p).$$

At $\Delta p = r$, we have $m_r(r) = \lambda u + (1 - \lambda) v = \frac{1}{2}$. Because F_r is strictly increasing, $m_r(\Delta p)$ is continuous and strictly decreasing in Δp, so the crossing is unique and equals r.

Applying the point-mass argument above to the model (B^*, ε_r) yields $\psi(B^*) = \psi(\delta_r)$ for every $r \in [c_1, c_2]$. Taking $r = c_1$ and $r = c_2$ gives

$$\psi(\delta_{c_1}) = \psi(B^*) = \psi(\delta_{c_2}).$$

Since $c_1 < c_2$ were arbitrary, it follows that $\psi(\delta_c) = K$ for some constant K.

Finally, take any model in $\mathcal{M}$, and let c denote its pooled crossing. The argument above gives $\psi(B) = \psi(\delta_c)$, while the preceding paragraph shows that $\psi(\delta_c) = K$ for every $c \in \mathbb{R}$. This implies $\psi(B) = K$ for every model in $\mathcal{M}$, i.e., that ψ is constant over $\mathcal{M}$ as claimed. $\square$

B Proofs for Section 3

B.1 Proof of Theorem 2

Proof. Fix $\tau \in (\alpha, 1-\alpha)$. We proceed in three steps.

The first step involves bounding quantiles of $B+\varepsilon$ on the observed price grid. We start by showing that for every $\nu \in (0,1)$,

$$\sup\{p_j : m(p_j) > 1-\nu\} \leq Q_{B+\varepsilon}(\nu) \leq \inf\{p_j : m(p_j) \leq 1-\nu\}.$$

To compute the lower bound, note that for any p_j satisfying $m(p_j) > 1-\nu$, we have

$$\begin{aligned}
\Pr(B+\varepsilon < p_j) &= 1 - \Pr(B+\varepsilon \geq p_j) \\
&= 1 - m(p_j) \\
&< \nu.
\end{aligned}$$

This implies $Q_{B+\varepsilon}(\nu) \geq p_j$. Taking the supremum over all such p_j gives

$$\sup\{p_j : m(p_j) > 1-\nu\} \leq Q_{B+\varepsilon}(\nu).$$

To compute the upper bound, note that for any p_j satisfying $m(p_j) \leq 1-\nu$, we have

$$\begin{aligned}
\Pr(B+\varepsilon \leq p_j) &= 1 - \Pr(B+\varepsilon > p_j) \\
&\geq 1 - m(p_j) \\
&\geq \nu.
\end{aligned}$$

This implies $Q_{B+\varepsilon}(\nu) \leq p_j$. Taking the infimum over all such p_j gives

$$Q_{B+\varepsilon}(\nu) \leq \inf\{p_j : m(p_j) \leq 1-\nu\}.$$

The second step derives inequalities linking the CDF of B to the CDF of $B+\varepsilon$. In particular, we show that for every $t \in \mathbb{R}$,

$$\Pr(B+\varepsilon \leq t-c) - \alpha \leq \Pr(B \leq t) \leq \Pr(B+\varepsilon \leq t+c) + \alpha.$$

For the upper bound, note that if $B \leq t$ and $\varepsilon \leq c$, then $B+\varepsilon \leq t+c$, so

$$\{B \leq t\} \subseteq \{B+\varepsilon \leq t+c\} \cup \{\varepsilon > c\}.$$

This implies

$$\begin{aligned}\Pr(B \le t) &\le \Pr(B + \varepsilon \le t + c) + \Pr(\varepsilon > c) \\ &\le \Pr(B + \varepsilon \le t + c) + \alpha. \end{aligned} \tag{1}$$

For the lower bound, note that if $B+\varepsilon \le t-c$ and $\varepsilon \ge -c$, then we have $B \le (t-c)+c = t$, so

$$\{B + \varepsilon \le t - c\} \cap \{\varepsilon \ge -c\} \subseteq \{B \le t\}.$$

This implies

$$\begin{aligned}\Pr(B \le t) &\ge \Pr(B + \varepsilon \le t - c) - \Pr(\varepsilon < -c) \\ &\ge \Pr(B + \varepsilon \le t - c) - \alpha. \end{aligned} \tag{2}$$

The third step involves converting the CDF inequalities in Equations (1) and (2) into quantile bounds for B. In particular, we will show that $Q_{B+\varepsilon}(\tau - \alpha) - c \le Q_B(\tau) \le Q_{B+\varepsilon}(\tau + \alpha) + c$.

To compute the lower bound, let $t < Q_{B+\varepsilon}(\tau - \alpha) - c$, so that $t + c < Q_{B+\varepsilon}(\tau - \alpha)$, which implies $\Pr(B + \varepsilon \le t + c) < \tau - \alpha$. By Equation (1), this gives

$$\Pr(B \le t) \le \Pr(B + \varepsilon \le t + c) + \alpha < \tau,$$

which implies $Q_B(\tau) > t$. Since this holds for every $t < Q_{B+\varepsilon}(\tau - \alpha) - c$, we obtain

$$Q_{B+\varepsilon}(\tau - \alpha) - c \le Q_B(\tau) \tag{3}$$

as desired.

To compute the upper bound, let $t = Q_{B+\varepsilon}(\tau + \alpha) + c$, so that $t - c = Q_{B+\varepsilon}(\tau + \alpha)$, which implies $\Pr(B + \varepsilon \le t - c) \ge \tau + \alpha$. By Equation (2), this gives

$$\Pr(B \le t) \ge \Pr(B + \varepsilon \le t - c) - \alpha \ge \tau,$$

which implies

$$Q_B(\tau) \le t = Q_{B+\varepsilon}(\tau + \alpha) + c \tag{4}$$

as desired.

Finally, applying the bounds from the first step with $\nu = \tau - \alpha$ and $\nu = \tau + \alpha$ gives

$$\begin{aligned}\sup\{p_j : m(p_j) > 1 - \tau + \alpha\} \leq Q_{B+\varepsilon}(\tau - \alpha) \\ Q_{B+\varepsilon}(\tau + \alpha) \leq \inf\{p_j : m(p_j) \leq 1 - \tau - \alpha\},\end{aligned}$$

which, combined with Equations (3) and (4), establishes the first claim.

Now suppose $B \perp \varepsilon$. For the upper bound, note that

$$\begin{aligned}\Pr(B + \varepsilon \leq t + c) &\geq \Pr(B \leq t, \varepsilon \leq c) \\ &= \Pr(B \leq t) \Pr(\varepsilon \leq c) \\ &\geq (1 - \alpha) \Pr(B \leq t).\end{aligned}$$

Thus

$$\Pr(B \leq t) \leq \frac{\Pr(B + \varepsilon \leq t + c)}{1 - \alpha}.$$

Similarly, for the lower bound, note that

$$\begin{aligned}\Pr(B + \varepsilon > t - c) &\geq \Pr(B > t, \varepsilon \geq -c) \\ &= \Pr(B > t) \Pr(\varepsilon \geq -c) \\ &\geq (1 - \alpha) \Pr(B > t).\end{aligned}$$

Equivalently,

$$\Pr(B \leq t) \geq \frac{\Pr(B + \varepsilon \leq t - c) - \alpha}{1 - \alpha}.$$

The rest of the argument is unchanged, with $\tau - \alpha$ replaced by $\tau(1 - \alpha)$ and $\tau + \alpha$ replaced by $\alpha + \tau(1 - \alpha)$.

If, in addition, $G(0) = \frac{1}{2}$, then

$$\Pr(\varepsilon \leq 0) = \frac{1}{2}$$

and

$$\Pr(\varepsilon \geq 0) \geq \frac{1}{2}.$$

For the upper bound, note that

$$\begin{aligned}\Pr(B+\varepsilon \le t) &\ge \Pr(B \le t, \varepsilon \le 0)\\ &= \Pr(B \le t)\Pr(\varepsilon \le 0)\\ &= \frac{1}{2}\Pr(B \le t).\end{aligned}$$

Thus

$$\Pr(B \le t) \le 2\Pr(B+\varepsilon \le t).$$

Similarly, for the lower bound, note that

$$\begin{aligned}\Pr(B+\varepsilon > t) &\ge \Pr(B > t, \varepsilon \ge 0)\\ &= \Pr(B > t)\Pr(\varepsilon \ge 0)\\ &\ge \frac{1}{2}\Pr(B > t).\end{aligned}$$

Equivalently,

$$\Pr(B \le t) \ge 2\Pr(B+\varepsilon \le t) - 1.$$

The rest of the argument is unchanged, with $\tau - \alpha$ replaced by $\frac{\tau}{2}$ for the lower bound and $\tau + \alpha$ replaced by $\frac{1+\tau}{2}$ for the upper bound. Combining these inequalities with the bound under independence gives the desired result. □

B.2 Proof of Theorem 3

Proof. Define $S_B(t) := \Pr(B \ge t)$ and $S_{B+\varepsilon}(t) := \Pr(B+\varepsilon \ge t)$. We first relate S_B to $S_{B+\varepsilon}$.

Note that, for every $t \in \mathbb{R}$, we have $\{B+\varepsilon \ge t+c\} \cap \{\varepsilon \le c\} \subseteq \{B \ge t\}$ because $B+\varepsilon \ge t+c$ and $\varepsilon \le c$ imply $B \ge (t+c)-c = t$. Thus

$$\begin{aligned}S_B(t) &= \Pr(B \ge t)\\ &\ge \Pr(B+\varepsilon \ge t+c) - \Pr(\varepsilon > c)\\ &\ge S_{B+\varepsilon}(t+c) - \alpha.\end{aligned}$$

Similarly, we have $\{B \ge t\} \subseteq \{B+\varepsilon \ge t-c\} \cup \{\varepsilon < -c\}$ because $B \ge t$ and $\varepsilon \ge -c$ imply $B+\varepsilon \ge t-c$. Thus

$$\begin{aligned}S_B(t) &\le \Pr(B+\varepsilon \ge t-c) + \Pr(\varepsilon < -c)\\ &\le S_{B+\varepsilon}(t-c) + \alpha.\end{aligned}$$

Because $S_B(t) \in [0,1]$, these inequalities can be sharpened to

$$S_B(t) \geq \max\{0, S_{B+\varepsilon}(t+c) - \alpha\} \tag{5}$$
$$S_B(t) \leq \min\{1, S_{B+\varepsilon}(t-c) + \alpha\}. \tag{6}$$

Next, note that for every realization of B,

$$B = \ell + \int_{\ell}^{u} \mathbb{1}_{\{B \geq t\}} \, \mathrm{d}t + (B-u)_+ - (\ell - B)_+.$$

Taking expectations gives

$$\mathbb{E}[B] = \ell + \int_{\ell}^{u} S_B(t) \, \mathrm{d}t + \mathbb{E}[(B-u)_+] - \mathbb{E}[(\ell - B)_+]. \tag{7}$$

Combining Equation (7) with Equation (5) gives

$$\begin{aligned} \mathbb{E}[B] &\geq \ell + \int_{\ell}^{u} \max\{0, S_{B+\varepsilon}(t+c) - \alpha\} \, \mathrm{d}t - \zeta \\ &= \ell + \int_{\ell+c}^{u+c} \max\{0, S_{B+\varepsilon}(s) - \alpha\} \, \mathrm{d}s - \zeta \\ &\geq \ell + \int_{\ell+c}^{u+c} \max\{0, \underline{m}(s) - \alpha\} \, \mathrm{d}s - \zeta, \end{aligned}$$

where the last inequality follows from the facts that $S_{B+\varepsilon}(s) \geq \underline{m}(s)$ for every s and that the function $x \mapsto \max\{0, x - \alpha\}$ is non-decreasing. Similarly, combining Equation (7) with Equation (6) gives

$$\begin{aligned} \mathbb{E}[B] &\leq \ell + \int_{\ell}^{u} \min\{1, S_{B+\varepsilon}(t-c) + \alpha\} \, \mathrm{d}t + \eta \\ &= \ell + \int_{\ell-c}^{u-c} \min\{1, S_{B+\varepsilon}(s) + \alpha\} \, \mathrm{d}s + \eta \\ &\leq \ell + \int_{\ell-c}^{u-c} \min\{1, \overline{m}(s) + \alpha\} \, \mathrm{d}s + \eta, \end{aligned}$$

where the last inequality follows from the facts that $S_{B+\varepsilon}(s) \leq \overline{m}(s)$ for every s and that the function $x \mapsto \min\{1, x + \alpha\}$ is non-decreasing. This establishes the first claim.

Now suppose $B \perp \varepsilon$. For the lower bound, note that $\{B < t, \varepsilon \leq c\} \subseteq \{B + \varepsilon < t + c\}$

implies

$$
\begin{aligned}
1 - S_{B+\varepsilon}(t+c) &\geq \Pr(B < t, \varepsilon \leq c) \\
&= \Pr(B < t)\Pr(\varepsilon \leq c) \\
&\geq (1 - S_B(t))(1-\alpha),
\end{aligned}
$$

and thus $S_B(t) \geq \frac{S_{B+\varepsilon}(t+c)-\alpha}{1-\alpha}$. For the upper bound, note that $\{B \geq t, \varepsilon \geq -c\} \subseteq \{B + \varepsilon \geq t - c\}$ implies

$$
\begin{aligned}
S_{B+\varepsilon}(t-c) &\geq \Pr(B \geq t, \varepsilon \geq -c) \\
&= S_B(t)\Pr(\varepsilon \geq -c) \\
&\geq S_B(t)(1-\alpha),
\end{aligned}
$$

and thus $S_B(t) \leq \frac{S_{B+\varepsilon}(t-c)}{1-\alpha}$. Therefore Equations (5) and (6) can be replaced by

$$
\begin{aligned}
S_B(t) &\geq \max\left\{0, \frac{S_{B+\varepsilon}(t+c)-\alpha}{1-\alpha}\right\}, \\
S_B(t) &\leq \min\left\{1, \frac{S_{B+\varepsilon}(t-c)}{1-\alpha}\right\}.
\end{aligned}
$$

The rest of the argument is unchanged, which establishes the bounds under independence.

In addition, if $G(0) = \frac{1}{2}$, then $\Pr(\varepsilon \leq 0) = \frac{1}{2}$ and $\Pr(\varepsilon \geq 0) \geq \frac{1}{2}$. For every $t \in \mathbb{R}$,

$$
\begin{aligned}
S_{B+\varepsilon}(t) &\geq \Pr(B \geq t, \varepsilon \geq 0) \\
&\geq \frac{1}{2}S_B(t),
\end{aligned}
$$

and hence $S_B(t) \leq 2S_{B+\varepsilon}(t)$. Similarly,

$$
\begin{aligned}
1 - S_{B+\varepsilon}(t) &= \Pr(B + \varepsilon < t) \\
&\geq \Pr(B < t, \varepsilon \leq 0) \\
&= \frac{1}{2}(1 - S_B(t)),
\end{aligned}
$$

and hence $S_B(t) \geq 2S_{B+\varepsilon}(t) - 1$. Combining these inequalities with the inequalities from above under independence alone gives

$$S_B(t) \geq \max\left\{0, \frac{S_{B+\varepsilon}(t+c) - \alpha}{1-\alpha}, 2S_{B+\varepsilon}(t) - 1\right\},$$
$$S_B(t) \leq \min\left\{1, \frac{S_{B+\varepsilon}(t-c)}{1-\alpha}, 2S_{B+\varepsilon}(t)\right\},$$

and the result follows. □

B.3 Proof of Corollary 1

Proof. If $\Pr(\ell \leq B \leq u) = 1$, then $(\ell - B)_+ = 0$ and $(B-u)_+ = 0$. Hence Theorem 3 applies with $\zeta = \eta = 0$.

For the case $|\varepsilon| \leq c$, note that $\alpha = 0$, so the tail restrictions reduce to $|\varepsilon| \leq c$, and the truncations at 0 and 1 do not bind. □

B.4 Proof of Theorem 4

Proof. Define $S_{B+\varepsilon}(t) := \Pr(B + \varepsilon \geq t)$. The assumption $\text{supp}(B+\varepsilon) \subseteq [L, U]$ implies

$$\begin{aligned}\mathbb{E}[B+\varepsilon] &= L + \mathbb{E}[B+\varepsilon-L] \\ &= L + \int_0^\infty \Pr(s \leq B + \varepsilon - L)\ \mathrm{d}s \\ &= L + \int_L^U S_{B+\varepsilon}(t)\ \mathrm{d}t,\end{aligned}$$

where the last equality follows from the change of variables $t = L + s$ and the fact that $S_{B+\varepsilon}(t) = 0$ for $t > U$. Since $\underline{m}(t) \leq S_{B+\varepsilon}(t) \leq \overline{m}(t)$ for every t, this implies

$$L + \int_L^U \underline{m}(t)\ \mathrm{d}t \leq \mathbb{E}[B+\varepsilon] \leq L + \int_L^U \overline{m}(t)\ \mathrm{d}t.$$

The bound on $\mathbb{E}[B+\varepsilon]$ also applies to $\mathbb{E}[B]$ since $\mathbb{E}[\varepsilon] = 0$ implies $\mathbb{E}[B] = \mathbb{E}[B+\varepsilon]$. □

B.5 Proof of Corollary 2

Proof of part 1. Suppose $\text{supp}(B+\varepsilon) \subseteq [L,U]$. By Theorem 4, the bound based directly on the support restriction for $B+\varepsilon$ has endpoints

$$B_L = L + \int_L^U \underline{m}(t)\ \mathrm{d}t, \tag{8}$$

$$B_U = L + \int_L^U \overline{m}(t)\ \mathrm{d}t. \tag{9}$$

Since $|\varepsilon| \le c$, this support restriction implies

$$\text{supp}(B) \subseteq [L-c, U+c].$$

Applying Corollary 1 with $\alpha = 0$, $\ell = L-c$, and $u = U+c$, the induced bound based on the support of B has endpoints

$$B_\ell = L - c + \int_L^{U+2c} \underline{m}(t)\ \mathrm{d}t, \tag{10}$$

$$B_u = L - c + \int_{L-2c}^{U} \overline{m}(t)\ \mathrm{d}t. \tag{11}$$

Because $\text{supp}(B+\varepsilon) \subseteq [L,U]$, we have $S_{B+\varepsilon}(t) = 0$ for $t > U$ and $S_{B+\varepsilon}(t) = 1$ for $t < L$. Since $\underline{m}(t) \le S_{B+\varepsilon}(t) \le \overline{m}(t)$ for every t, we must have $\underline{m}(t) = 0$ for $t > U$ and $\overline{m}(t) = 1$ for $t < L$. Therefore

$$\int_L^{U+2c} \underline{m}(t)\ \mathrm{d}t = \int_L^U \underline{m}(t)\ \mathrm{d}t \tag{12}$$

and

$$\begin{aligned}\int_{L-2c}^{U} \overline{m}(t)\ \mathrm{d}t &= \int_{L-2c}^{L} 1\ \mathrm{d}t + \int_L^U \overline{m}(t)\ \mathrm{d}t \\ &= 2c + \int_L^U \overline{m}(t)\ \mathrm{d}t.\end{aligned} \tag{13}$$

Substituting Equation (12) into Equation (10) and then applying Equation (8) gives

$$B_\ell = L - c + \int_L^U \underline{m}(t)\ \mathrm{d}t = B_L - c.$$

Likewise, substituting Equation (13) into Equation (11) and then applying Equation (9) gives

$$B_u = L - c + 2c + \int_L^U \overline{m}(t)\ \mathrm{d}t = B_U + c.$$

This implies

$$B_\ell \le B_L \le B_U \le B_u,$$

so the bound $[B_L, B_U]$ is weakly tighter than the induced bound $[B_\ell, B_u]$. □

Proof of part 2. Suppose $\text{supp}(B) \subseteq [\ell, u]$. By Corollary 1 with $\alpha = 0$, the bound based directly on the support restriction for B has endpoints

$$\begin{aligned} B_\ell &= \ell + \int_{\ell+c}^{u+c} \underline{m}(t)\ \mathrm{d}t \\ B_u &= \ell + \int_{\ell-c}^{u-c} \overline{m}(t)\ \mathrm{d}t. \end{aligned}$$

Since $|\varepsilon| \le c$, this support restriction implies

$$\text{supp}(B + \varepsilon) \subseteq [\ell - c, u + c].$$

Applying Theorem 4 with $L = \ell - c$ and $U = u + c$, the induced bound based on the support of $B + \varepsilon$ has endpoints

$$\begin{aligned} B_L &= \ell - c + \int_{\ell-c}^{u+c} \underline{m}(t)\ \mathrm{d}t \\ B_U &= \ell - c + \int_{\ell-c}^{u+c} \overline{m}(t)\ \mathrm{d}t. \end{aligned}$$

Subtracting the lower endpoints gives

$$\begin{aligned} B_L - B_\ell &= \left[\ell - c + \int_{\ell-c}^{u+c} \underline{m}(t)\ \mathrm{d}t\right] - \left[\ell + \int_{\ell+c}^{u+c} \underline{m}(t)\ \mathrm{d}t\right] \\ &= -c + \int_{\ell-c}^{\ell+c} \underline{m}(t)\ \mathrm{d}t. \end{aligned} \tag{14}$$

Similarly, subtracting the upper endpoints gives

$$\begin{aligned} B_U - B_u &= \left[\ell - c + \int_{\ell-c}^{u+c} \overline{m}(t)\ \mathrm{d}t\right] - \left[\ell + \int_{\ell-c}^{u-c} \overline{m}(t)\ \mathrm{d}t\right] \\ &= -c + \int_{u-c}^{u+c} \overline{m}(t)\ \mathrm{d}t. \end{aligned} \tag{15}$$

Since $0 \leq \underline{m}(t) \leq 1$ and $0 \leq \overline{m}(t) \leq 1$ for every t, the integrals from Equations (14) and (15) fall in the interval $[0, 2c]$, and thus

$$\begin{aligned} B_L - B_\ell &\in [-c, c] \\ B_U - B_u &\in [-c, c]. \end{aligned}$$

As the three-price example in the main text illustrates, these endpoint differences can be positive or negative. Thus neither bound is uniformly contained in the other. □

B.6 Theorem B.1 and its proof

Theorem B.1. *If $m_a(\cdot)$ is known on a finite set of observed prices for some attribute-difference vector a, then the bounds from Theorems 2 to 4 apply directly to $\mathbb{E}[a^\top W]$ and $Q_{a^\top W}(\tau)$ after replacing B by $a^\top W$ and m by m_a.*

Furthermore, for any such collection of attribute-difference vectors $a_1, \ldots, a_J$ associated with bounds $L_j \leq \mathbb{E}\left[a_j^\top W\right] \leq U_j$, the mean WTP vector $\mu := \mathbb{E}[W]$ satisfies

$$\mu \in \bigcap_{j=1}^{J} \left\{\mu \in \mathbb{R}^d : L_j \leq a_j^\top \mu \leq U_j\right\}.$$

Proof. Conditional on $\Delta x_{it} = a$, the choice rule becomes $y_{it} = \mathbb{1}_{\{a^\top W_i - \Delta p_{it} + \varepsilon_{it} \geq 0\}}$. The pooled choice probability at price p is then given by $m_a(p) = \mathbb{E}[F(a^\top W - p)] = \Pr(a^\top W + \varepsilon \geq p)$. Since this is exactly the same form as the single-attribute model with B replaced by $a^\top W$ and m replaced by m_a, the quantile and mean bounds apply directly.

For the final claim, since $\mathbb{E}\left[a_j^\top W\right] = a_j^\top \mathbb{E}[W] = a_j^\top \mu$ for any j, intersecting the bounds $L_j \leq a_j^\top \mu \leq U_j$ over $j = 1, \ldots, J$ gives the desired result. □

C Proofs for Section 5

C.1 Proof of Lemma 1

Proof. Let the design μ place weight $w_a, w_b > 0$ on a and b, with $w_a + w_b = 1$. Define

$$q_c(\beta, \sigma) := F\left(\frac{\beta - c}{\sigma}\right)$$

for $c \in \{a, b\}$. In this case, Equation (4) becomes

$$\begin{aligned} Q_\mu(\beta, \sigma) = w_a[p_a \log q_a(\beta, \sigma) + (1 - p_a) \log(1 - q_a(\beta, \sigma))] \\ + w_b[p_b \log q_b(\beta, \sigma) + (1 - p_b) \log(1 - q_b(\beta, \sigma))]. \end{aligned}$$

Note that the function $\phi_p(q) := p \log q + (1 - p) \log(1 - q)$ with $p, q \in (0, 1)$ is strictly concave and uniquely maximized at $q = p$. This implies that $Q_\mu(\beta, \sigma)$ is uniquely maximized when

$$q_a(\beta, \sigma) = p_a \tag{16}$$

$$q_b(\beta, \sigma) = p_b. \tag{17}$$

Next we show that there exists a unique (β, σ) with $\sigma > 0$ satisfying these two equations. Since F is the logistic CDF, Equations (16) and (17) are equivalent to

$$\frac{\beta - a}{\sigma} = \ell(p_a) \tag{18}$$

$$\frac{\beta - b}{\sigma} = \ell(p_b). \tag{19}$$

Subtracting these yields

$$\sigma^*(\mu) = \frac{b - a}{\ell(p_a) - \ell(p_b)},$$

which is positive because $p_a > p_b$ (since $a < b$) and $\ell(\cdot)$ is strictly increasing. Finally, substituting $\sigma^*(\mu)$ into Equation (18) and Equation (19) yields

$$\begin{aligned}
\beta^*(\mu) &= b + \sigma^*(\mu)\ell(p_b) \\
&= b + \frac{b-a}{\ell(p_a)-\ell(p_b)}\ell(p_b), \\
\beta^*(\mu) &= a + \sigma^*(\mu)\ell(p_a) \\
&= a + \frac{b-a}{\ell(p_a)-\ell(p_b)}\ell(p_a),
\end{aligned}$$

which establishes the claimed formulas for $\sigma^*(\mu)$ and $\beta^*(\mu)$. □

C.2 Proof of Theorem 5

Proof of part 1. Suppose $p_b > \frac{1}{2}$. Then $\ell(p_b) > 0$, so Equation (7) from Lemma 1 implies $\beta^*(\mu) > b$ as well as $\beta^*(\mu) > \beta_H$ if and only if

$$b + \frac{b-a}{\ell(p_a)-\ell(p_b)}\ell(p_b) > \beta_H.$$

Since $a < b$ implies $\ell(p_a) - \ell(p_b) > 0$, this is equivalent to

$$(b-a)\ell(p_b) > (\beta_H - b)(\ell(p_a) - \ell(p_b)),$$

or

$$(\beta_H - a)\ell(p_b) > (\beta_H - b)\ell(p_a).$$

Dividing both sides by $(\beta_H - b)\ell(p_b)$ gives

$$\frac{\ell(p_a)}{\ell(p_b)} < \frac{\beta_H - a}{\beta_H - b}.$$

□

Proof of part 2. Suppose $p_a < \frac{1}{2}$. Then $\ell(p_a) < 0$, so Equation (8) from Lemma 1 implies $\beta^*(\mu) < a$ as well as $\beta^*(\mu) < \beta_L$ if and only if

$$a + \frac{b-a}{\ell(p_a)-\ell(p_b)}\ell(p_a) < \beta_L.$$

Since $p_a > p_b$ implies $\ell(p_a) - \ell(p_b) > 0$, this is equivalent to

$$(b-a)\ell(p_a) < (\beta_L - a)(\ell(p_a) - \ell(p_b)),$$

or

$$(b - \beta_L)\ell(p_a) < (a - \beta_L)\ell(p_b).$$

Dividing both sides by $(b - \beta_L)\ell(p_b)$ reverses the inequality because $b > \beta_L$ and $p_b < p_a < \frac{1}{2}$ imply $(b - \beta_L)\ell(p_b) < 0$. Therefore

$$\frac{\ell(p_a)}{\ell(p_b)} > \frac{a - \beta_L}{b - \beta_L}.$$

□

C.3 Proof of Corollary 3

Proof. Define $a = \beta_L + t\sigma$ and $b = \beta_H - t\sigma$. If $\sigma < \frac{\beta_H - \beta_L}{2t}$, then $a < b$, and both points lie in $[\beta_L, \beta_H]$.

At these design points,

$$m(a) = \pi F(-t) + (1 - \pi)F\left(\frac{\beta_H - \beta_L}{\sigma} - t\right)$$
$$m(b) = \pi F\left(t - \frac{\beta_H - \beta_L}{\sigma}\right) + (1 - \pi)F(t).$$

As $\frac{\beta_H - \beta_L}{\sigma} \to \infty$, we have

$$m(a) \to 1 - \pi + \pi F(-t),$$
$$m(b) \to (1 - \pi)F(t).$$

By assumption, $(1 - \pi)F(t) > \frac{1}{2}$, so both limits lie in $(\frac{1}{2}, 1)$. The limiting value of $m(a)$ exceeds the limiting value of $m(b)$ since

$$[1 - \pi + \pi F(-t)] - [(1 - \pi)F(t)] = 1 - F(t) = F(-t) > 0.$$

Thus there exists $K_1 > 2t$ such that for all $\frac{\beta_H - \beta_L}{\sigma} > K_1$,

$$\frac{1}{2} < m(b) < m(a) < 1.$$

Since $m(a)$ and $m(b)$ converge to strictly positive limits in $(\frac{1}{2}, 1)$, $\frac{\ell(m(a))}{\ell(m(b))}$ also converges to a finite positive constant.

Moreover,

$$\frac{\beta_H - a}{\beta_H - b} = \frac{(\beta_H - \beta_L) - t\sigma}{t\sigma} = \frac{\beta_H - \beta_L}{t\sigma} - 1.$$

This expression diverges to infinity as $\frac{\beta_H - \beta_L}{\sigma} \to \infty$. Therefore there exists $K_2 \geq K_1$ such that for all $\frac{\beta_H - \beta_L}{\sigma} > K_2$,

$$\frac{\ell(m(a))}{\ell(m(b))} < \frac{\beta_H - a}{\beta_H - b}.$$

By Theorem 5, this implies $\beta^*(\mu) > \beta_H$, so setting $K = K_2$ completes the proof. □

C.4 Proof of Theorem 6

Proof. Fix $\pi \in (0, 1)$, $\beta_0 \in \mathbb{R}$, and $M > 0$. Let

$$m(\Delta p) = \pi F\left(\frac{\beta_0 - \Delta p}{\sigma_L}\right) + (1 - \pi) F\left(\frac{\beta_0 - \Delta p}{\sigma_H}\right)$$

for $\sigma_H > \sigma_L > 0$. Define

$$\kappa := \frac{1}{\sigma_L} - \frac{1}{\sigma_H},$$
$$L := \log(1 - \pi).$$

Since $\sigma_H > \sigma_L$, we have $\kappa > 0$.

We first analyze the left-tail design. Let μ_R^- be supported on

$$a_R^- := \beta_0 - R - 1,$$
$$b_R^- := \beta_0 - R.$$

Using $1 - F(x) = \frac{e^{-x}}{1+e^{-x}}$, we can write

$$1 - m(a_R^-) = (1 - \pi) e^{-\frac{R+1}{\sigma_H}} x_R, \tag{20}$$
$$1 - m(b_R^-) = (1 - \pi) e^{-\frac{R}{\sigma_H}} y_R, \tag{21}$$

where

$$x_R := \frac{1}{1 + e^{-\frac{R+1}{\sigma_H}}} + \frac{\pi}{1 - \pi} \frac{e^{-(R+1)\kappa}}{1 + e^{-\frac{R+1}{\sigma_L}}},$$
$$y_R := \frac{1}{1 + e^{-\frac{R}{\sigma_H}}} + \frac{\pi}{1 - \pi} \frac{e^{-R\kappa}}{1 + e^{-\frac{R}{\sigma_L}}}.$$

Therefore $x_R \to 1$ and $y_R \to 1$. It also follows from Equations (20) and (21) that $m(a_R^-) \to 1$ and $m(b_R^-) \to 1$.

Define

$$\alpha_R := \log m(a_R^-) - \log x_R,$$
$$\beta_R := \log m(b_R^-) - \log y_R.$$

Since $m(a_R^-)$, $m(b_R^-)$, x_R, and y_R all converge to 1, we have $\alpha_R \to 0$ and $\beta_R \to 0$. Moreover, these terms converge to 0 exponentially fast, so

$$R|\alpha_R| + R|\beta_R| \to 0. \tag{22}$$

Using $\ell(u) = \log u - \log(1-u)$ together with Equations (20) and (21), we obtain

$$\ell(m(a_R^-)) = \frac{R+1}{\sigma_H} - L + \alpha_R, \tag{23}$$
$$\ell(m(b_R^-)) = \frac{R}{\sigma_H} - L + \beta_R. \tag{24}$$

By Lemma 1, and because $b_R^- - a_R^- = 1$,

$$\beta^*(\mu_R^-) = b_R^- + \frac{\ell(m(b_R^-))}{\ell(m(a_R^-)) - \ell(m(b_R^-))}. \tag{25}$$

Substituting Equations (23) and (24) and $b_R^- = \beta_0 - R$ into Equation (25) gives

$$\beta^*(\mu_R^-) = \beta_0 - R + \frac{\frac{R}{\sigma_H} - L + \beta_R}{\frac{1}{\sigma_H} + \alpha_R - \beta_R}. \tag{26}$$

Subtracting $\beta_0 - \sigma_H L$ from Equation (26) and simplifying gives

$$\beta^*(\mu_R^-) - (\beta_0 - \sigma_H L) = \frac{\beta_R - (R - \sigma_H L)(\alpha_R - \beta_R)}{\frac{1}{\sigma_H} + \alpha_R - \beta_R}. \tag{27}$$

The denominator in Equation (27) converges to $\frac{1}{\sigma_H} > 0$. The numerator converges to 0 by Equation (22). Therefore

$$\beta^*(\mu_R^-) \to \beta_0 - \sigma_H \log(1-\pi). \tag{28}$$

We now analyze the right-tail design. Let μ_R^+ be supported on

$$a_R^+ := \beta_0 + R,$$
$$b_R^+ := \beta_0 + R + 1.$$

Using $F(-x) = \frac{e^{-x}}{1+e^{-x}}$, we can write

$$m(a_R^+) = (1-\pi)e^{-\frac{R}{\sigma_H}} u_R, \tag{29}$$
$$m(b_R^+) = (1-\pi)e^{-\frac{R+1}{\sigma_H}} v_R, \tag{30}$$

where

$$u_R := \frac{1}{1+e^{-\frac{R}{\sigma_H}}} + \frac{\pi}{1-\pi}\frac{e^{-R\kappa}}{1+e^{-\frac{R}{\sigma_L}}},$$
$$v_R := \frac{1}{1+e^{-\frac{R+1}{\sigma_H}}} + \frac{\pi}{1-\pi}\frac{e^{-(R+1)\kappa}}{1+e^{-\frac{R+1}{\sigma_L}}}.$$

Therefore $u_R \to 1$ and $v_R \to 1$. It also follows from Equations (29) and (30) that $m(a_R^+) \to 0$ and $m(b_R^+) \to 0$.

Define

$$\gamma_R := \log u_R - \log(1 - m(a_R^+)),$$
$$\delta_R := \log v_R - \log(1 - m(b_R^+)).$$

Then $\gamma_R \to 0$ and $\delta_R \to 0$. Moreover,

$$R|\gamma_R| + R|\delta_R| \to 0. \tag{31}$$

Using $\ell(u) = \log u - \log(1-u)$ together with Equations (29) and (30), we obtain

$$\ell(m(a_R^+)) = L - \frac{R}{\sigma_H} + \gamma_R, \tag{32}$$
$$\ell(m(b_R^+)) = L - \frac{R+1}{\sigma_H} + \delta_R. \tag{33}$$

By Lemma 1, and because $b_R^+ - a_R^+ = 1$,

$$\beta^*(\mu_R^+) = b_R^+ + \frac{\ell(m(b_R^+))}{\ell(m(a_R^+)) - \ell(m(b_R^+))}. \tag{34}$$

Substituting Equations (32) and (33) and $b_R^+ = \beta_0 + R + 1$ into Equation (34) gives

$$\beta^*(\mu_R^+) = \beta_0 + R + 1 + \frac{L - \frac{R+1}{\sigma_H} + \delta_R}{\frac{1}{\sigma_H} + \gamma_R - \delta_R}. \tag{35}$$

Subtracting $\beta_0 + \sigma_H L$ from Equation (35) and simplifying gives

$$\beta^*(\mu_R^+) - (\beta_0 + \sigma_H L) = \frac{\delta_R - (R + 1 + \sigma_H L)(\gamma_R - \delta_R)}{\frac{1}{\sigma_H} + \gamma_R - \delta_R}. \tag{36}$$

The denominator in Equation (36) converges to $\frac{1}{\sigma_H} > 0$. The numerator converges to 0 by Equation (31). Therefore

$$\beta^*(\mu_R^+) \to \beta_0 + \sigma_H \log(1 - \pi). \tag{37}$$

Since $\log(1 - \pi) < 0$, choose $\sigma_H > \sigma_L > 0$ large enough so that

$$-\sigma_H \log(1 - \pi) > M + 1.$$

Then

$$\begin{aligned} \beta_0 - \sigma_H \log(1 - \pi) &> \beta_0 + M + 1 \\ \beta_0 + \sigma_H \log(1 - \pi) &< \beta_0 - M - 1. \end{aligned}$$

By Equations (28) and (37), we can choose R large enough so that

$$\left|\beta^*(\mu_R^-) - (\beta_0 - \sigma_H \log(1 - \pi))\right| < 1$$

and

$$\left|\beta^*(\mu_R^+) - (\beta_0 + \sigma_H \log(1 - \pi))\right| < 1.$$

This implies $\beta^*(\mu_R^-) > \beta_0 + M$ and $\beta^*(\mu_R^+) < \beta_0 - M$, and hence $\left|\beta^*(\mu_R^-) - \beta^*(\mu_R^+)\right| > 2M$. □

References

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